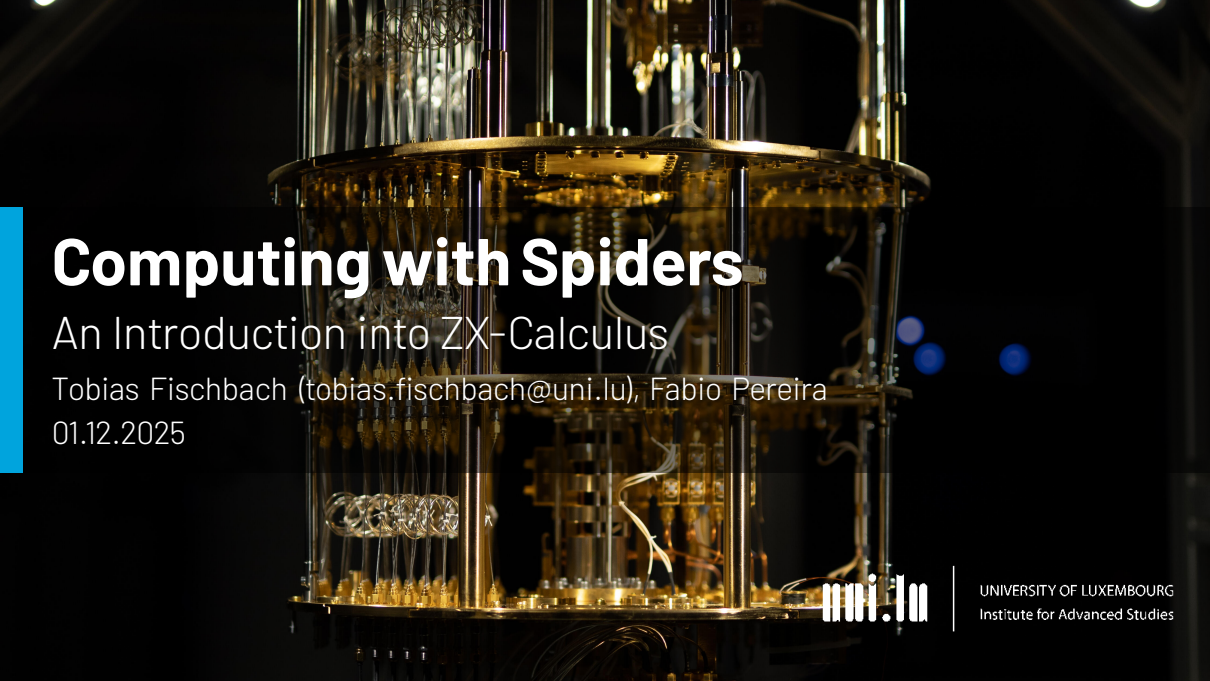

Computing with Spiders

An Introduction into ZX-Calculus

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01.12.2025

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What is a Quantum Program?

Linear Map of Qubits (Ex. Single-Qubit)

- Let T be a function $T : \mathcal{H} \to \mathcal{H}$
- Let $\mathcal{H}$ be the complex Hilbert space $\mathcal{H} = \mathbb{C}^2$ of a single-qubit
- For $|\psi\rangle, |\phi\rangle \in \mathcal{H}, \forall \alpha, \beta \in \mathbb{C}$:
  - Scalar multiplication: $T(\alpha |\psi\rangle) = \alpha T(|\psi\rangle)$
  - Addivity: $T(\alpha |\psi\rangle + \beta |\phi\rangle) = \alpha T(|\psi\rangle) + \beta T(|\phi\rangle)$

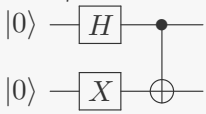
Modern Quantum Mechanics

- *"Matrizenmechanik"* (Heisenberg)
- Bohr, Born and Heisenberg gave rise to modern quantum theory (*Copenhagen interpretation*)
  - Uncertainty relation
  - Operator representation
  - Commutator relation
  - Correspondence principle

Figure: Werner Heisenberg, 1929

Quantum Circuits

Example: Bell state



$$|0\rangle \otimes |0\rangle = |00\rangle$$

$$CX\dot{(}H \otimes X)\,|00\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^+\rangle$$

Example: Not-Not Cancellation

$$|0\rangle - \boxed{X} - \boxed{X} - = \ldots$$

$$\hat{X}\hat{X}\,|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} |0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} |0\rangle = |0\rangle$$

ZX-Calculus

ZX-Calculus [3] is a diagrammatic reasoning framework that is:

- Independent of a given universal gate set
- Universal
- Compact semantic-preserving rule set
- Complete

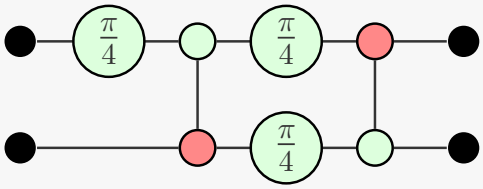
Good introduction: *ZX-calculus for the Working Quantum Computer Scientist* [27]

ZX-Diagram

Labelled Open-Graph

Definition

- Let $G(\mathbf{V}, \mathbf{E}, \mathbf{I}, \mathbf{O}, \alpha, \beta)$ be a *finite undirected graph*
- Let $\mathbf{V}$ be the set of vertices and $\mathbf{E}$ the set of edges
- Let $\mathbf{I} \subseteq \mathbf{V}$ be the set of inputs and $\mathbf{O} \subseteq \mathbf{V}$ the set of outputs
  - Inputs and outputs only consist of single-degree vertices $v \in \mathbf{I} \cup \mathbf{O}, \deg(v) = 1$
  - $\mathbf{I} \cup \mathbf{O}$ forms the *boundary* of G
- Let $\mathbf{S} = \mathbf{V} \setminus (\mathbf{I} \cup \mathbf{O})$ be the set of interior vertices called *spiders*
- Let $\alpha : \mathbf{S} \to \{Z, X\}$ be a labelling function that assigns a spiders's *basis*
- Let $\beta : \mathbf{S} \to \{\frac{n\pi}{4} | n \in \mathbb{Z}\}$ be a labelling function that assigns a spider's *phase* (here Clifford + T)



Generators

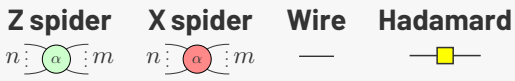


Figure: Generators of ZX calculus.

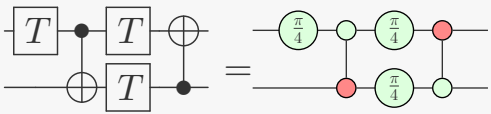


Figure: Quantum Circuit to ZX diagram conversion

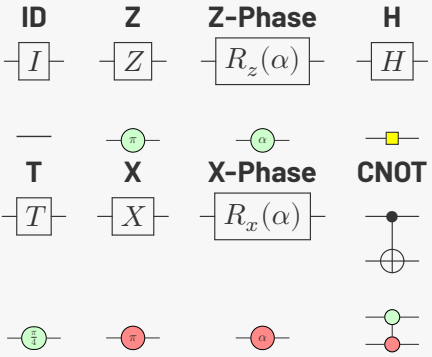
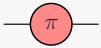
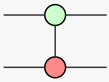


Figure: Translation from gates to generators.

Quantum Gates

Gate	*Unitary Matrix*	*Quantum Circuit*	*ZX-Calculus*	*Hermitian*	*Single-Qubit*
Identity	$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$			☑	☑
Pauli-X	$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$			☑	☑
Hadamard	$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$			☑	☑
CNOT	$CX = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$			☑	☒ (2-qubit)

Definition

- Let **ZX** be the set of all labeled open graphs
- Let **LM** be the set of linear maps between qubits
- Let $\gamma : \mathbf{ZX} \to \mathbf{LM}$ be a function that maps ZX-diagrams to its linear map of qubits
- Two ZX-diagrams have the same semantics if they admit the same linear map of qubits $g, h \in \mathbf{ZX}, \gamma(g) = \gamma(h)$
- Let a rewriting rule be a function $r : \mathbf{ZX} \to \mathbf{ZX}$:
  - that transforms a ZX-diagram
  - *AND* preserves the semantics, such that $g \in \mathbf{ZX}, \gamma(g) = (\gamma \circ r)(g)$.

Rewriting Rules

Examples

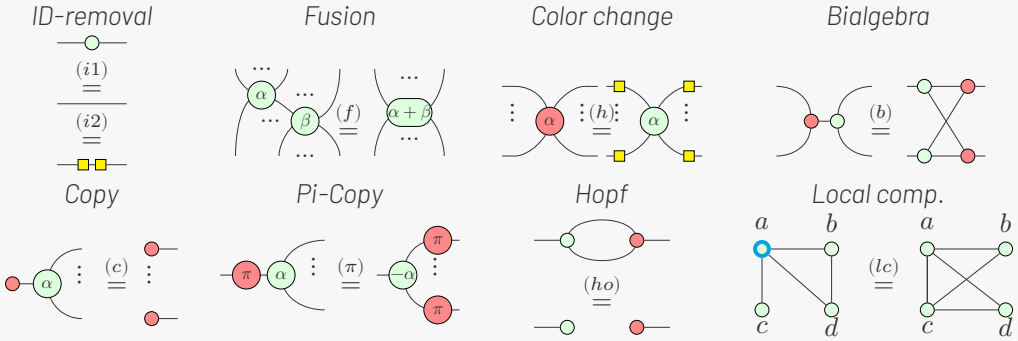


Figure: The basic rewriting rules of ZX-calculus. [4, 25]

Circuit Extraction

Why is it so tough?

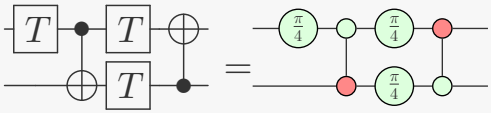


Figure: ZX-diagrams do not have a clear order of execution

- Circuit extraction is #P-hard [9]
- No flow of time, no clear order of execution
- Circuit extraction is equivalent to finding a unique measurement-pattern (generalised flow)

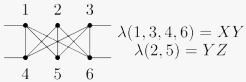
Circuit Extraction

Generalised Flow [1]

A labelled open graph (G, I, O, λ) has gflow if there exists a map $g : \hat{O} \to P(\hat{I})$ and a partial order $\prec$ over V such that $\forall v \in \hat{O}$:

- if $w \in g(v)$ and $v\neg = w$, $v \prec w$
- if $w \in Odd(g(v))$ and $v\neg = w$, then $v \prec w$
- if $\lambda(v) = XY$, then $v \notin g(v)$ and $v \in Odd(g(v))$
- if $\lambda(v) = XZ$, then $v \in g(v)$ and $v \in Odd(g(v))$
- if $\lambda(v) = YZ$, then $v \notin g(v)$ and $v \notin Odd(g(v))$

$g(v)$ is the correction set of v.



Here the conditions are fulfilled with the following map:

v	1	2	4	5
$g(v)$	$\{5\}$	$\{2,3\}$	$\{2\}$	$\{5,6\}$
$Odd\,(g(v))$	$\{1,2,3\}$	$\{\}$	$\{4,5,6\}$	$\{\}$

and the ordering

$$1, 4 \prec 2, 5 \prec 3, 6.$$

Circuit Extraction

Graphlike ZX-Diagrams

Definition

- only Z-spiders
- only Hadamard wires
- Input / Outputs can only connect to one spider

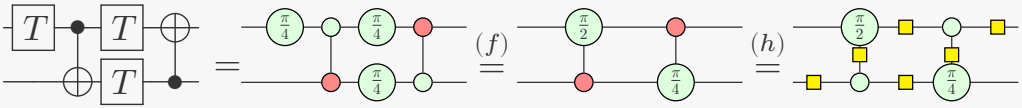
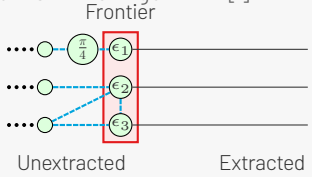


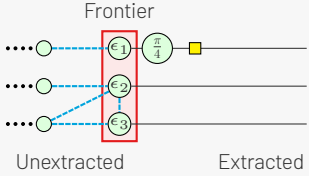
Figure: Every ZX-diagram can be converted to a graph-like diagram using the h- and i-rules.

Circuit Extraction

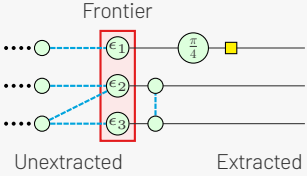
Polynomial-time Algorithm [1]



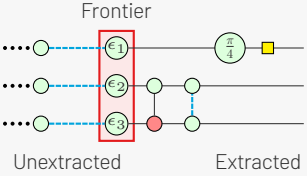
(a) Unextracted ZX-diagram.



(b) Spiders with a single input and output can be extracted directly.



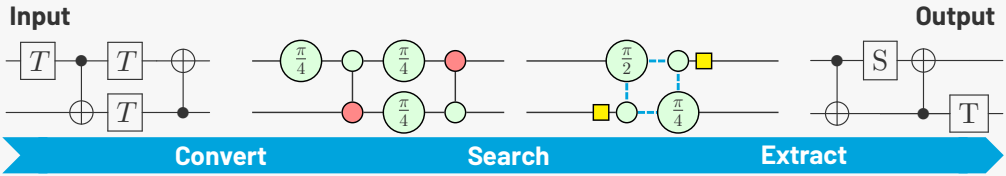
(c) Connections between frontier spiders are extracted by CZ gates.



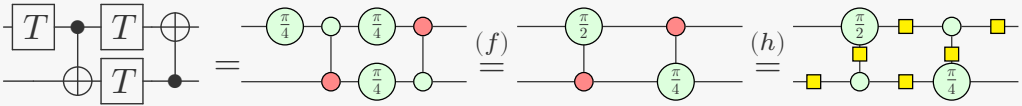
(d) CNOT gates extract wires between frontier and unextracted spiders.

ZX-Diagram Optimization

Pipeline



Example



ZX-Diagram Optimization

Overview

Author(s)	*Year*	*T*	*2Q*	*Depth*	*Qubits*	*Edges*	*Vertices*
Fagan & Duncan [12]	2019		y	y			
Kissinger et al. [17]	2019		y				
Beaudrap et al. [8]	2020	y					
Beaudrap et al. [7]	2020	y					
Duncan et al. [10]	2020	y					
Kissinger & Wetering [18]	2020	y					
Cowtan et al. [5]	2020		y				
Cowtan et al. [6]	2020		y				
Munson et al. [22]	2021	y					
Zilk et al. [31]	2022	y	y	y			
Gogioso & Yeung [14]	2023		y				
Staudacher et al. [25]	2023		y			y	
Winderl et al. [30]	2023		y				
Riu et al. [24]	2023		y			y	
Griend et al. [21]	2023		y				
Vandaele [28]	2024				y		
Holker [15]	2024	y				y	
Nägele & Marquardt [23]	2024						y
Staudacher et al. [26]	2024			y			
Ewen et al. [11]	2025		y	y			
Huang et al. [16]	2025	y	y				
Mattick et al. [20]	2025		y			y	
Fischbach et al. [13]	2025	y				y	
Liu et al. [19]	2024			y	y		
Chen et al. [2]	2025		y	y			
Villoria et al. [29]	2025		y				

ZX-Benchmark

- Benchmarking framework of current ZX-based optimizers
- Integrates with QASM
- Currently implemented:
  - Holker 2024
  - Staudacher 2023
  - Ewen 2025
  - Fischbach 2025

Figure: Please test!

Take-Home Message

- Every quantum circuit can be converted into a ZX-diagram
- Circuit extraction is a combinatorial problem that is #P-hard
- Compact semantic-preserving rule set
- *Intermediate representation of Quantum Computing*

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